Rainbow connections for planar graphs and line graphs*

Xiaolong Huang, Xueliang Li, Yongtang Shi
Center for Combinatorics and LPMC-TJKLC
Nankai University, Tianjin 300071, China
Email: xiaolonghuang@ymail.com, lxl@nankai.edu.cn, shi@nankai.edu.cn

Abstract

An edge-colored graph G is rainbow connected if any two vertices are connected by a path whose edges have distinct colors. The rainbow connection number of a connected graph G, denoted by $rc(G)$, is the smallest number of colors that are needed in order to make G rainbow connected. It was proved that computing $rc(G)$ is an NP-Hard problem, as well as that even deciding whether a graph has $rc(G) = 2$ is NP-Complete. It is known that deciding whether a given edge-colored graph is rainbow connected is NP-Complete. We will prove that it is still NP-Complete even when the edge-colored graph is a planar bipartite graph. We also give upper bounds of the rainbow connection number of outerplanar graphs with small diameters. A vertex-colored graph is rainbow vertex-connected if any two vertices are connected by a path whose internal vertices have distinct colors. The rainbow vertex-connection number of a connected graph G, denoted by $rvc(G)$, is the smallest number of colors that are needed in order to make G rainbow vertex-connected. It is known that deciding whether a given vertex-colored graph is rainbow vertex-connected is NP-Complete. We will prove that it is still NP-Complete even when the vertex-colored graph is a line graph.

Keywords: computational complexity; rainbow connection; coloring; planar graph; line graph

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1 Introduction

All graphs considered here are simple, finite and undirected. We follow the notation and terminology of [20]. An edge-colored graph is *rainbow connected* if any two vertices are connected by a path whose edges have distinct colors (such paths are called *rainbow path*). Obviously, if G is rainbow connected, then it is also connected. This concept of rainbow connection in graphs was introduced by Chartrand et al. in [5]. The rainbow connection number of a connected graph G, denoted by $rc(G)$, is the smallest number of colors that are needed in order to make G rainbow connected. Observe that $diam(G) \leq rc(G) \leq n-1$. It is easy to verify that $rc(G) = 1$ if and only if G is a complete graph, that $rc(G) = n-1$ if and only if G is a tree. Chartrand et al. computed the precise rainbow connection number of several graph classes including complete multipartite graphs ([5]). The rainbow connection number has been studied for further graph classes in [2, 9, 14, 15] and for graphs with fixed minimum degree in [2, 10, 18]. There are also some results on the aspect of extremal graph theory, such as [19]. Very recently, many results on the rainbow connection have been reported in a survey of Li and Sun [17].

Besides its theoretical interest as a natural combinatorial concept, rainbow connection has an interesting application for the secure transfer of classified information between agencies ([8]). While the information needs to be protected, there must also be procedures that permit access between appropriate parties. This twofold issue can be addressed by assigning information transfer paths between agencies which may have other agencies as intermediaries, while requiring a large enough number of passwords and firewalls that is prohibitive to intruders, yet small enough to manage (that is, enough that one or more paths between every pair of agencies have no password repeated). An immediate question arises: what is the minimum number of passwords or firewalls needed that allows one or more secure paths between every two agencies such that the passwords along each path are distinct?

The complexity of determining the rainbow connection of a graph has been studied in literature. It is proved that the computation of $rc(G)$ is NP-hard [3, 11]. In fact it is already NP-complete to decide whether $rc(G) = 2$, and in fact it is already NP-complete to decide whether a given edge-colored (with an unbounded number of colors) graph is rainbow connected [3]. More generally it has been shown in [11] that for any fixed $k \geq 2$, deciding whether $rc(G) = k$ is NP-complete. Moreover, the authors in [13] proved that it is still NP-Complete even when the edge-colored graph is bipartite. Ananth and Nasre [1] showed that for any fixed integer $k \geq 3$, deciding whether $rc(G) = k$ is NP-Complete.

In this paper, we will prove that it is still NP-Complete to decide whether a given edge-colored graph is rainbow connected even when the edge-colored graph is a planar bipartite graph. As deciding whether $rc(G) = 2$ is NP-complete, the authors in [7, 12] considered bridgeless graphs with diameter two and proved that the rainbow connection number in this case can not exceed 5. We will show that the rainbow connection number is at most three for bridgeless outerplanar graphs with diameter two, and at most six for bridgeless outerplanar graphs with diameter three.

A vertex-colored graph is *rainbow vertex-connected* if any two vertices are connected by a path whose internal vertices have distinct colors (such paths are called *vertex rainbow path*). The *rainbow vertex-connection* of a connected graph G, denoted by $rvc(G)$, is the smallest number of colors that are needed in order to make G rainbow vertex-connected. An easy observation is that if G is of order n then $rvc(G) \leq n - 2$ and $rvc(G) = 0$ if and only if G is a complete graph. Notice that $rvc(G) \geq diam(G) - 1$ with equality if the diameter is 1 or 2. For rainbow connection and rainbow vertex-connection, some examples are given to show that there is no upper bound for one of parameters in terms of the other in [10]. The rainbow vertex-connection number has been studied for graphs with fixed minimum degree in [10, 16]. In [6], Chen, Li and Shi studied the complexity of determining the rainbow vertex-connection of a graph and prove that computing $rvc(G)$ is NP-Hard. Moreover, they proved that it is already NP-Complete to decide whether $rvc(G) = 2$. They also proved that it is already NP-complete to decide whether a given vertex-colored graph is rainbow vertex-connected. In this paper, we will prove that it is still NP-Complete to decide whether a given vertex-colored graph is rainbow vertex-connected even when the vertex-colored graph is a line graph.

2 Rainbow connection for planar graphs

Before proceeding, we list some related results as useful lemmas.

Lemma 1 ([3]) *The following problem is NP-Complete: Given an edge-colored graph G, check whether the given coloring makes G rainbow connected.*

By subdividing each edge of a given edge-colored graph G exactly once, one can get a bipartite graph G'. Then color the edges of G' as follows: Let e' and e'' be the two edges of G' produced by subdividing at the edge e of G. Then color the edge e' with the same color of e and color the edge e'' with a new color, such that all the new colors of the edges e'' are distinct. In this way, Li and Li proved the following result from the problem in Lemma 1.

Lemma 2 ([13]) *Given an edge-colored bipartite graph G, checking whether the given coloring makes G rainbow connected is NP-Complete.*

A *plane graph* is a planar graph together with an embedding of the graph in the plane. From the Jordan Closed Curve Theorem, we know that a cycle C in a plane graph separates the plane into two regions, the interior of C and the exterior of C. We prove the following result.

Theorem 1 *Given an edge-colored planar graph G, checking whether the given coloring makes G rainbow connected is NP-Complete.*

Proof. By Lemma 1, it will suffice by showing a polynomial reduction from the problem in Lemma 1.

Given a graph $G = (V, E)$ and an edge-coloring c of G, we will construct an edge-colored planar graph G' such that G is rainbow connected if and only if G' is rainbow connected.

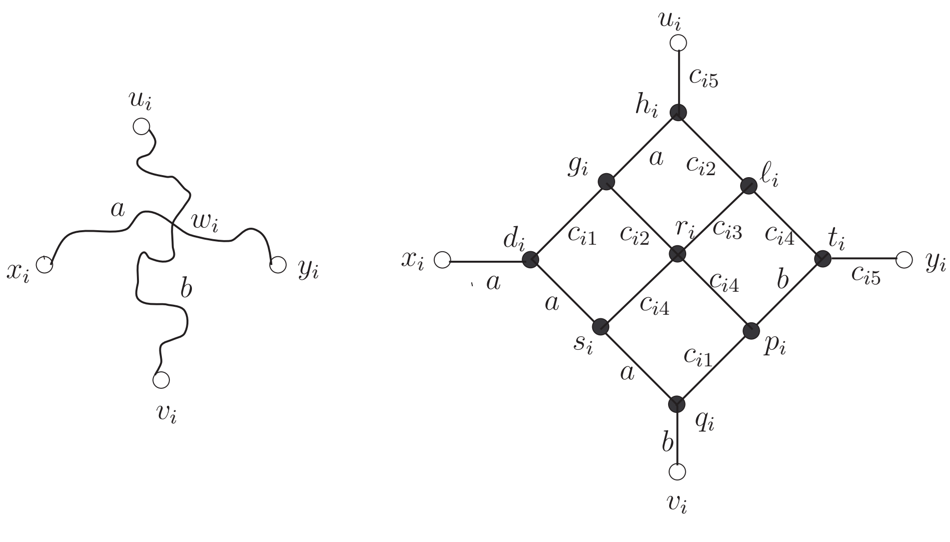

Figure 1: The graph constructed in Theorem 1 for some crossing w_i.

For one drawing of a given graph, by moving edges slightly, we can ensure that no three edges have a common crossing and no two edges cross more than once. Given a such drawing of G in the plane with k crossings, denoted by w_i, where $i = 1, 2, \ldots, k$. Let w_i be formed by two edges x_iy_i and u_iv_i. First, we assume that there is at most one crossing on each edge.

We construct an edge-colored graph G' as follows. Graph $G' = (V', E')$ is obtained from G by replacing each crossing w_i with one 3×3-grid with

vertex set $\{d_i,\ g_i,\ h_i,\ \ell_i,\ r_i,\ s_i,\ t_i,\ p_i,\ q_i\}$, as shown in Figure 1. Therefore, we have $V' = V \cup \{d_i,\ g_i,\ h_i,\ \ell_i,\ r_i,\ s_i,\ t_i,\ p_i, q_i : 1 \leq i \leq k\}$, $E' = E \cup \{x_id_i,\ y_it_i,\ u_ih_i,\ v_iq_i,\ d_ig_i,\ g_ih_i,\ h_i\ell_i,\ g_ir_i,\ d_is_i,\ \ell_ir_i,\ r_is_i,\ \ell_it_i,\ r_ip_i,\ s_iq_i,\ p_iq_i,\ p_it_i : 1 \leq i \leq k\}$. From our construction, we know that G' is planar. In the following, we define an edge-coloring c' of G': $c'(e) = c(e)$ for each $e \in E$; $c'(x_id_i) = c'(d_is_i) = c'(s_iq_i) = c'(g_ih_i) = c(x_iy_i)$, $c'(v_iq_i) = c'(p_it_i) = c(u_iv_i)$, $c'(d_ig_i) = c'(p_iq_i) = c_{i1}$, $c'(g_ir_i) = c'(h_i\ell_i) = c_{i2}$, $c'(r_i\ell_i) = c_{i3}$, $c'(r_ip_i) = c'(\ell_it_i) = c'(r_is_i) = c_{i4}$, $c'(u_ih_i) = c'(t_iy_i) = c_{i5}$, where c_{ij} are the new colors for $1 \leq i \leq k$ and $1 \leq j \leq 5$.

Suppose coloring c' makes G' rainbow connected. For any two vertices $u, v \in V$, there is a rainbow path P' connected u and v. If P' does not pass any grid, then P' is also a rainbow path joining u and v in G under the coloring c. Otherwise, suppose P' passes some grid. We give the following claim.

Claim. If the rainbow path P' enters to a grid from vertex x_i (or y_i), then it must be go out from y_i (or x_i).

Notice that $x_id_ig_ir_i\ell_it_iy_i$ is a rainbow path enters to the grid from x_i to y_i. From the definition of c', one can easily to check that there has no rainbow path from x_i (or y_i) to u_i and v_i, which just go through this grid. ∎

Similarly, one also can prove that if the rainbow path P' enters to a grid from vertex u_i (or v_i), then it must be go out from v_i (or u_i). Denote by $P'(x_i, y_i)$ ($P'(u_i, v_i)$) the subpath joining vertices x_i and y_i (u_i and v_i) in path P' and let P'' be the path obtained from P' by deleting $P'(x_i, y_i)$ ($P'(u_i, v_i)$) and adding edge x_iy_i (u_iv_i). Applying this operation for each grid appeared in path P' yields one path P of G, which is also a rainbow path in G under the coloring c. It follows that the coloring c makes G rainbow connected.

To prove the other direction, suppose the coloring c makes G rainbow connected. Let u and v be a pair of vertices in G'. We will find a rainbow path joining u and v in G' under the coloring c' and then obtain that c' makes G rainbow connection.

Case 1. $u, v \in V$.

If there is a rainbow path joining u and v without going through any crossing, then this path is also a rainbow path joining u and v in G' under the coloring c'. Now let P be the rainbow path joining u and v and some crossing w_i lies on P. Without loss of generality, suppose $P = u \ldots x_iy_i \ldots v$. Then the new path P' obtained from P by replacing the edge x_iy_i with path $x_id_ig_ir_i\ell_it_iy_i$ is the required rainbow path joining u and v in G'.

Case 2. $u, v \in \{d_i,\ g_i,\ h_i,\ \ell_i,\ r_i,\ s_i,\ t_i,\ p_i,\ q_i : 1 \leq i \leq k\}$, i.e., u and v belongs to the same grid.

In this case, one can easily to find a rainbow path connecting u and v from the definition of c'.

Case 3. $u \in V$, $v \in \{d_i,\ g_i,\ h_i,\ \ell_i,\ r_i,\ s_i,\ t_i,\ p_i, q_i : 1 \leq i \leq k\}$.

It is easy to find the required rainbow path for the case of $u = u_i$ or $u = y_i$. Now suppose $u \notin \{u_i, v_i\}$. Since there exists a rainbow path P' joining u and u_i (or y_i) in G' by **Case 1**, attaching the rainbow path between u_i (or y_i) and v to P' yields the required rainbow path connecting u and v.

Case 4. u and v belongs to the different grids.

From the above cases, the proof of this case is obviously.

In any case, there exists one rainbow path connecting u and v in G' under the coloring c'.

Notice that this reduction is indeed a polynomial reduction, since each graph has at most $\binom{n}{2}$ crossings and for each crossing, we introduce nine vertices, fourteen edges and five new colors in the construction of graph G'.

Suppose there are more than one crossings on some edge e, we can add one vertex with degree two between any two distinct crossings on the same edge and then assign color $c(e)$ and a new color c_1 to the two new edges. Since each graph has at most $\binom{n}{2}$ crossings, we may introduce at most $\binom{n}{2}$ new vertices and $\binom{n}{2}$ new colors. Similarly, we can complete the polynomial reduction. $\square$

Using the same subdividing method for reducing Lemma 1 to Lemma 2, we can get the following corollary.

Corollary 1 *Given an edge-colored planar bipartite graph G, checking whether the given coloring makes G rainbow connected is NP-Complete.*

We now consider a more restricted class of planar graphs, namely outerplanar graphs. A planar graph G is said to be *outerplanar* if G can be embedded in the plane in such a way that all vertices are incident with a common face. From this it is easy to see that any 2-connected outerplanar graph has a Hamilton cycle. Now we give a property of outerplanar graphs.

Proposition 1 ([20]) *Every simple outerplanar graph has a vertex of degree at most two.*

In [14], the authors proved that $rc(G) \leq \lceil n/2 \rceil$ for any 2-connected graph G. Since $rc(C_n) \leq \lceil n/2 \rceil$, where C_n denotes the cycle graph of order n, we can deduce that $rc(G) \leq \lceil n/2 \rceil$ for any Hamiltonian graph G.

Proposition 2 *Let G be a Hamiltonian graph, then $rc(G) \leq \lceil n/2 \rceil$.*

As the rainbow connection number is at least the diameter and deciding whether $rc(G) = 2$ is NP-complete, it is necessary to determine the rainbow

connection number of graphs with diameter two. The authors in [7, 12] considered bridgeless graphs with diameter two and proved that the rainbow connection number in this case can not exceed 5.

Lemma 3 ([7, 12]) *If G is a connected bridgeless graph with diameter 2, then $rc(G) \leq 5$. Moreover, the upper bound is sharp.*

We show that the rainbow connection number is at most three for bridgeless outerplanar graphs with diameter two, and at most six for bridgeless outerplanar graphs with diameter three.

A subset D of the vertices in G is called a *dominating set* if every vertex of $G - D$ is adjacent to a vertex of D. Furthermore, if the dominating set D induces a connect subgraph of G, then D is called a *connected dominating set*. Let $X, Y \in V(G)$, we say that X *dominates* Y if every vertex of Y is adjacent to at least one vertex of X. The following lemma will be used in the sequent.

Lemma 4 ([4]) *For any connected graph G with minimum degree at least two. Let D be a connected dominating set of G, then $rc(G) \leq rc(G[D]) + 3$.*

Theorem 2 *If G is a bridgeless outerplanar graph with order n and diameter two, then $rc(G) \leq 3$, i.e., $rc(G) = 2, 3$.*

Proof. Suppose that G is a bridgeless outerplanar graph with diameter two. If G has a cut vertex, then this vertex is a domination set of the graph, then $rc(G) \leq 3$. Now we suppose that G is 2-connected and we can embed G so that a Hamilton cycle, H, bounds the outer face, and the edges not in H are chords that lie in the interior of H. If G has no chords, then G is a cycle of length at most five and thus $rc(G) \leq 3$. In the following we assume G has chords. Let v be a vertex with degree two and suppose $N(v) = \{x_1, y_1\}$. Denote by C the induced cycle of G containing vertex v. We will consider the following two cases according to the order of C.

Case 1. $|C| = 4$.

Suppose $C = vx_1zy_1v$. In this case, there are at most two vertices outside of C, since each vertex outside of C must be adjacent to both x_1 and z (or y_1 and z). Observe that $rc(G) = 2$.

Case 2. $|C| = 3$.

For convenience, we assume $H = vx_1x_2 \dots x_{n/2}y_{(n-2)/2} \dots y_2y_1v$ for even n and $H = vx_1x_2 \dots x_{(n-1)/2}y_{(n-1)/2} \dots y_2y_1v$ for odd n. If H has only one chord, then this case is the same as **Case 1**. Otherwise, H has at least two chords and then $n \geq 5$. There must be one chord e such that one of its end vertex is x_1 or y_1, without loss of generality, say x_1. Then, the other end

of e must be y_2 or x_3. Assume $e = x_1y_2$, then all other vertices in the set $V \setminus \{v, x_1, x_2, y_1, y_2\}$ must be adjacent to x_1, as the diameter of G is two. Therefore, in this case, the structure of graph G is a fan, as shown in Figure 2.

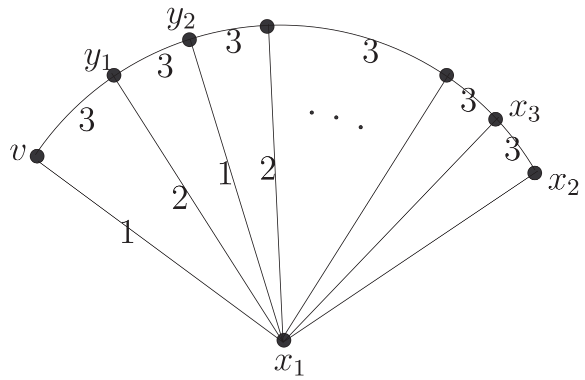

Figure 2: Bridgeless outerplanar graph with diameter two.

For $n = 5$, we can give an edge coloring c of G such that $rc(G) = 2$ under this coloring: $c(vx_1) = c(vy_1) = c(x_1y_2) = c(x_2y_2) = 1$ and $c(x_1y_1) = c(y_1y_2) = c(x_1x_2) = 2$. For $n \geq 6$, we observe that $rc(G) = 3$. Notice that 2 colors cannot make G rainbow connected. Now we give one edge coloring with three colors: all edges with x_1 as one of its end are assigned colors 1 and 2 alternatively in clockwise order; all other edges are assigned color 3, as shown in Figure 2. $\square$

Theorem 3 *If G is a bridgeless outerplanar graph with order n and diameter three, then $3 \leq rc(G) \leq 6$.*

Proof. Suppose that $G = (V, E)$ is a bridgeless outerplanar graph with diameter three. Since the rainbow connection number is at least the diameter, then we have $rc(G) \geq 3$. Suppose G is not 2-connected and let v be a cut vertex of G. There is a partition of $V - \{v\}$ into two sets A and B such that vertex v dominates either A or B. Without loss of generality, we may assume v dominates A. Denote by B_1 the vertices of B that are adjacent to v and $B_2 = B - B_1$. Choose a minimum cardinality subset S of B_1 such that S dominates B_2. Then $S \cup \{v\}$ is a connected dominating set. We claim that $|S| \leq 2$. Suppose that $|S| \geq 3$ and let $S = \{s_1, s_2, s_3\}$. By the minimality of S, there exist three vertices $x_1, x_2, x_3 \in B_2$ satisfying that among three vertices s_1, s_2, s_3, x_i is only adjacent to vertex s_i for $1 \leq i \leq 3$. Take $a \in A$.

Without loss of generality, we may assume that an embedding of G as an outerplanar graph has vertices a, s_1, s_2, s_3 in clockwise order, adjacent to v. Since all the vertices of G lie on a common face, there is no way to obtain a path of length at most three between x_1 and x_3. Thus, $|S| \leq 2$, which yields that $rc(G) \leq rc(G[S \cup \{v\}]) + 3 = 5$.

Now suppose G is 2-connected. It follows that G can be embedded in such a way that a Hamilton cycle H bounds the outer face, and the edges not in H are chords that lie in the interior of H. If H has no chords, then G is a cycle of length at most seven, and thus $rc(G) = 3$ or $rc(G) = 4$. Thus, in the following we assume H has at least one chord.

Suppose xy is a chord of H. Cycle H is divided into two xy-paths. We denote the path goes in clockwise direction from x to y by the xy-segment of H, and denote the other path by the yx-segment of H.

Now suppose H has precisely one chord xy. In this case, $\{x, y\}$ is a vertex cut of G. Since G has diameter three, then $\{x, y\}$ dominates either xy-segment of H or the yx-segment of H. Without loss of generality, we suppose that xy-segment is dominated. Since there are no other chords, the xy-segment of H is a path of length two or three. If it is two, then the yx-segment of H is a path of length four or five and thus we can check that $rc(G) = 3$. Otherwise, the yx-segment of H is a path of length three or four and thus $rc(G) = 3$ or 4.

Suppose H has at least two chords. Among all vertex cuts with two vertices, we choose $\{a, b\}$ as a vertex cut such that it dominates a maximum number of vertices. Note that a and b may not correspond to the ends of a chord of H. Since G has diameter three, $\{a, b\}$ dominates one segment of H. Without loss of generality, we assume ba-segment of H is dominated by $\{a, b\}$. Consider the ab-segment of H.

Case 1. There are no chords with both ends on the ab-segment of H.

In this case, there are at least two chords in the ba-segment. It follows that there are at most three internal vertices in the ab-segment of H. Now we suppose there are three internal vertices in the ab-segment of H, since it is easy to check that $rc(G) \leq 6$ for the other two cases. If $ab \in E(G)$, then there exists a connected dominating set with three vertices and then $rc(G) \leq 5$. Otherwise, we claim that there exists a vertex v in the ba-segment such that $va,\ vb \in E(G)$, since G has diameter 3 and at least two chords. It implies that G has a connected dominating set with four vertices, then $rc(G) \leq 6$.

Case 2. There are some chords with both ends on the ab-segment of H.

Choose a vertex cut of size two, $\{c, d\}$, such that any other vertex cut of size two with both vertices in the ab-segment of H has at least one vertex in the cd-segment of H, where the cd-segment is a part of the ab-segment.

Subcase 2.1. a, b, c, d are not all distinct vertices.

Without loss of generality, we suppose $b = d$. By our choice, any vertex on the ac-segment of H does not form of a vertex cut with b, and hence ac must be an edge of G (ac may be an edge of H or a chord of H).

Suppose $\{c, b\}$ can not dominate the cb-segment. Let v be a vertex on the vc-segment such that $d(v, c) \geq 2$ and $d(v, b) \geq 2$. Then all vertices in ba-segment must be adjacent to vertex b. Therefore, all vertices in ac-segment must be adjacent to vertex c, since otherwise, if there exists a vertex w such that $wa \in E(G)$ and $wc \notin E(G)$, then $d(w, v) \geq 4$. Thus, $\{b, c\}$ is a vertex cut with two vertices, which dominates more vertices than $\{a, b\}$, a contradiction to the choice of $\{a, b\}$.

Now suppose $\{c, b\}$ dominates the cb-segment. Thus, $\{a, b, c\}$ must be a dominating set of G. If one of ab and bc is an edge of G, then $\{a, b, c\}$ is a connected dominating set of G and thus $rc(G) \leq 2 + 3 = 5$. Now suppose neither ab nor bc is an edge of G.

Subsubcase 2.1.1. There is vertex v in ba-segment (or cb-segment) such that v is adjacent to both a and b (or c and b).

In this situation, $\{a, b, c, v\}$ is a connected dominating set of G and thus $rc(G) \leq 3 + 3 = 6$.

Subsubcase 2.1.2. Otherwise, there does not exist such vertex.

Each vertex in ba-segment is only adjacent to one of a and b, and each vertex in cb-segment is only adjacent to one of c and b. Now in this case, each of ba-segment and cb-segment of H has at least two internal vertices. We claim each of ba-segment and cb-segment of H has exactly two internal vertices, since otherwise, we always can find two vertices with distance at least four. Since G has at least two chords, then we can assume that ac-segment has at least two internal vertices, which also implies one pair of vertices with distance at least four.

Subcase 2.2. a, b, c, d are distinct vertices.

The choice of $\{c, d\}$ implies that neither ad nor bc is an edge of G. From the way that $\{a, b\}$ and $\{c, d\}$ was chosen, we know that $\{a, b\}$ dominates the ba-segment and $\{c, d\}$ dominates the dc-segment. Moreover, ac and bd must be edges of G. If there is one vertex p in ba-segment such that it is adjacent to a but not adjacent to b, and also one vertex q in cd-segment such that it is adjacent to d but not adjacent to c, then $d(p, q) \geq 4$, a contradiction. Therefore, either $\{a, b, c\}$ or $\{b, c, d\}$ is a dominating set of G. We assume that $\{a, b, c\}$ is a dominating set of G, since the other case is similar. If ab is an edge of G, then $\{a, b, c\}$ is a connected dominating set of G and thus $rc(G) \leq 2 + 3 = 5$. Now suppose ab is not an edge of G.

Subsubcase 2.2.1. There is vertex v in ba-segment such that v is adjacent to both a and b.

In this situation, $\{a, b, c, v\}$ is a connected dominating set of G and thus $rc(G) \leq 3 + 3 = 6$.

Subsubcase 2.2.2. Otherwise, there does not exist such vertex.

In this situation, each vertex in the cd-segment must be adjacent to both c and d, which implies that cd-segment contains exactly one internal vertex. Similarly, there exactly two internal vertices in the ba-segment. Since G has two chords, then there are some internal vertices in the ac-segment and bd-segment. In each case, we can find two vertices with distance at least four.

The proof is thus completed.

□

3 Rainbow vertex-connection for line graphs

In [6], the complexity of determining the rainbow vertex-connection of a graph has been studied. The following result was proved.

Lemma 5 ([6]) *The following problem is NP-Complete: given a vertex-colored graph G, check whether the given coloring makes G rainbow vertex-connected.*

We will prove that it is still NP-Complete to decide whether a given vertex-colored graph is rainbow connected even when the vertex-colored graph is a line graph.

Theorem 4 *The following problem is NP-Complete: given a vertex-colored line graph G, check whether the given coloring makes G rainbow vertex-connected.*

Proof. By Lemma 1, it will suffice by showing a polynomial reduction from the problem in Lemma 1.

Given a graph $G = (V, E)$ and an edge-coloring c of G. We want to construct a line graph G' with a vertex coloring such that G' is rainbow vertex-connected iff G is rainbow connected.

Let $G = (V, E)$ and suppose $V = \{v_1, x_2, \ldots, v_n\}$ and $E = \{e_1, e_2, \ldots, e_m\}$. Let $G_0 = (V_0, E_0)$ be a new graph, which is obtained from G by attaching a pendent vertex u_i to v_i for each $1 \leq i \leq n$. Thus, $V_0 = V \cup \{u_1, u_2, \ldots, u_n\}$ and $E_0 = E \cup \{e'_i = u_iv_i : 1 \leq i \leq n\}$. Let G' be the line graph of G_0 and then $V(G') = E_0$. Now we define a vertex coloring c' as follows: for each $1 \leq i \leq n$, $c'(e_i) = c(e_i)$ and $c'(e'_i) = c_0$, where c_0 is a new color we introduced.

Suppose G is rainbow connected under the edge coloring c, then we will check that there exists one vertex rainbow path between any pair of vertices in G' under the vertex coloring c'. Consider the pair of e'_i and e'_j for $i \neq j$. Let $v_{i_0}v_{i_1}\ldots v_{i_{k+1}}$ be the rainbow path between v_i and v_j in G, where $v_{i_0} = v_i$ and $v_{i_{k+1}} = v_j$. Denote by $e_{i_t} = v_{i_t}v_{i_{t+1}}$ for $0 \leq t \leq k$. Thus, we have that edges $e_{i_0},\ e_{i_1},\ \ldots,\ e_{i_k}$ have distinct colors. By the definition of c', the colors of vertices $e_{i_0},\ e_{i_1},\ \ldots,\ e_{i_k}$ in G' are all distinct. Thus, $e'_ie_{i_0}e_{i_1}\ldots e_{i_k}e'_j$ is a required vertex rainbow path. Similarly, for the pair e'_i and e_j, and the pair e_i and e_j, we can find vertex rainbow paths in G', respectively.

Now suppose G' is rainbow vertex-connected under the vertex coloring c', then we will check that there exists one rainbow path between any pair of vertices in G under the coloring c. For each pair e'_i and e'_j, where $1 \leq i \neq j \leq n$, there exists one vertex rainbow path $e'_ie_{i_0}e_{i_1}\ldots e_{i_{k+1}}e'_j$, i.e., $e_{i_0},\ e_{i_1},\ \ldots,\ e_{i_{k+1}}$ has distinct colors. Observe that in G, one of end vertices of e_{i_0} is v_i and one of end vertices of $e_{i_{k+1}}$ is v_j. Thus, there indeed exists one rainbow path connecting v_i and v_j.

The proof is thus completed.

$\square$

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